Euler Trails: Leader Instructions

Once upon a time there was a city called Königsburg that had a branching river, an island, and seven bridges. The people of Königsburg decided to try to find a way to design a walking tour that would cross each bridge exactly once and then return to the starting place, but no one could do it. Girls will try several puzzles of this type and will figure out the mystery of the bridges of Königsburg. They will apply what they learn to figure out when it is possible to hike on all of the trails in a camp exactly once.

Levels Brownies, Juniors, Cadettes, and Seniors

Topics Graph Theory, Problem Solving, Number Patterns

Goals

- Girls will learn what *graph theory* is.
- Girls will learn what *nodes* and *edges* are.
- Girls will learn the definition of an *Euler trail.*
- Girls will learn that the *degree* of a node is the number of edges coming into or out of it.
- Girls will learn how to tell when a graph has an Euler trail and when it does not.

Preparation Time 30 minutes if building or drawing maps; none otherwise

Activity Time 25 to 35 minutes

Materials and Preparation

- Sidewalk chalk and pavement OR branches and open space OR tape on a floor OR handouts and pencils for working with maps
- Each group needs 10 to 20 small objects OR chalk OR string to mark paths.
- If using sidewalk chalk or branches, draw or build at least one map for every two to four girls.

Primary Source University of Alabama Center for Teaching and Learning: Math 103. "Euler Paths and Circuits." http://www.ctl.ua.edu/math103/

Background

Allow 5 minutes for the background discussion. Tell the girls that they will be working in an area of mathematics called *graph theory.* Graph theory is the study of how different places are connected to one another. Graph theorists study links between websites on the internet, find efficient methods for shipping things to stores, and plan airline flights.

There was once a city in East Prussia called Königsburg. (Today, Königsberg is called Kaliningrad and is located in Russia.) In the 1750s, Königsburg had seven bridges as shown on the handout. Many people tried to design a walking route that would cross each bridge exactly once. No one could solve the problem.

When Leonhard Euler (pronounced "oiler") heard about the challenge, he examined lots of maps. He realized that it is sometimes impossible to walk on each path exactly once. For example, see the first map on the back of the *Euler Trails* handout. This map shows a central hub with four paths leading out to four different locations. Why is it not possible to walk on each path exactly once on this map?

Graphs have *nodes* that are connected by *edges.* In the bridges of Königsberg problem, the nodes are the four different pieces of land. The edges are the bridges.

A graph has an *Euler trail* if there is a route that uses each path exactly once. You may visit the same place more than once, but you can't use a path again once you have walked on it. You are never allowed to go part way down a path and then turn around. The trail does not have to end where it began.

Activity Instructions

Allow 10 to 20 minutes for girls to try the different maps and decide which ones have Euler trails. Girls should work in teams of 2 to 4 people. As the girls are working on the maps, ask them to think about why some maps have an Euler trail and why others do not. What is the difference between them?

If you have a small number of main trails in your camp, the girls can see whether the camp has an Euler trail. Including smaller trails may change the answer.

In the maps on the back of the handout, each location is represented as a large dot or circle. Euler found it helpful to use graphs to represent maps where the land is more spread out. For example, for the map of Königsburg, Euler represented one side of the river with a node labeled A, the other side of the river with node C, the island in the middle with node B, and the land between the two rivers with node D. He connected the nodes with edges to represent the bridges. The graph for each Königsberg/Kaliningrad map is shown next to it on the handout. You may wish to show the girls how to build this type of graph to represent maps where the land is more spread out.

Conclusion

Allow 10 minutes for the concluding discussion.

When everyone has had a chance to try several maps, gather everyone together. For each map, ask whether anyone found an Euler trail. Ask the girls whether it matters which node they start at or end at. (Sometimes it does, sometimes it doesn't.) What is special about the nodes they must start or end at? If they need more help, ask them to count how many edges come out of each node on the graph.

Girls should eventually notice that nodes with an odd number of edges are trickier to work with compared to nodes with an even number of edges. If a node has an odd number of edges then it must either be the starting or ending place. Ask the girls why they think that is true.

The *degree* of a node is the number of edges coming into or out of the node. Nodes that have an odd degree must either be the starting or ending place for an Euler trail if one exists. If a node with an odd degree were not the starting or ending place, then we must use one of the trails to arrive at the node. This leaves an even number of edges that have not been used. To use all of those edges it would be necessary to leave then come back to the node, leave then come back to the node. Each trip out and back uses up two edges and brings us back to the same node. If this node is not the ending node, then we would either get stuck at the node or have no way to use up all the edges.

Since a node with an odd number of edges must either be a starting or ending place, then any graph with more than two nodes of odd degree cannot have an Euler trail. If the graph has no nodes of odd degree or exactly two nodes of odd degree then it is possible to find an Euler trail. (It is not possible for a graph to have only one node of odd degree – try it!)

Look at each map as a group and ask girls to find the nodes of odd degree. In particular, notice that for the original Bridges of Königsberg problem, all four vertices have odd degree. This is why the problem was impossible to solve.